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1980 J. Phys. A: Math. Gen. 13 1507

(http://iopscience.iop.org/0305-4470/13/4/037)

J. Phys. A: Math. Gen. 13 (1980) 1507–1515. Printed in Great Britain

General discrete planar models in two dimensions: duality properties and phase diagrams†

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Received 20 August 1979

Abstract. We consider the most general spin model with nearest-neighbour interactions invariant under a global Z_p symmetry, in two dimensions. Dual transformations are discussed, and the subset of self-dual models is characterised. The phase diagrams for $p \geqslant 5$ are particularly rich, containing first-order, second-order and infinite-order phase transitions. In particular, the existence of a massless phase similar to the low-temperature phase of the XY model is established.

1. Introduction

Recently, it has been realised that the peculiar properties of the O(2) planar model (classical XY model) in two dimensions imply an interesting structure for the case in which the O(2) symmetry is broken down to that of a Z_p subgroup. José *et al* (1977) showed that Z_p perturbations on the O(2) model are irrelevant for a range of temperatures below the critical temperature T_c of the O(2) model, provided $p \geqslant 5$. Thus, these models should undergo two phase transitions as a function of temperature. For $T < T_{c1}$, the Z_p symmetry is broken and the correlation length is finite. For $T_{c1} < T < T_{c2}$ the symmetry is unbroken and the correlation length is infinite (the 'massless' behaviour characteristic of the low-temperature phase of the O(2) model), while for $T > T_{c2}$ we have an unbroken phase with short-range order. Elitzur *et al* (1979) extended these results to the 'pure' Z_p case when the Z_p perturbations become large, and the angular variables become discretised in units of $2\pi/p$. These authors considered mainly the Villain (1975) form of the interaction, which lends itself to simple duality transformations and Griffiths-type inequalities (Griffiths 1972, Ginibre 1970).

In this paper we consider the most general form of a 'pure' Z_p theory (that is, the angles are discretised) with nearest-neighbour interactions. Thus, each site i of a square lattice is occupied by an angular variable θ_i, a multiple of $2\pi/p$. The nearest-neighbour interaction $V(\theta_i - \theta_j)$ can thus take on $[p/2]+1$ values‡. Since only energy differences are relevant, this means that our theory will possess $[p/2]$ parameters. Fortunately, the interesting physical cases probably correspond to $p \leqslant 6$, and so this proliferation is not too serious. For these models, it is possible to construct duality transformations which are generalisations of the original transformations of Kramers and Wannier (1941) for the case $p = 2$ and of Potts (1952) for the eponymous model, which corresponds to the case $V(\theta_i - \theta_j) \propto \delta_{\theta_i \theta_j}$. These turn out to be nothing more than Fourier transforms in the

† Work supported by National Science Foundation.
‡ We denote the integral part of a real number a by $[a]$.

space $\{x_r\}$, where x_r: exp $V(2\pi r/p)$. This linear realisation enables us to make several general deductions concerning the nature of self-dual Z_p models. In particular, we show that there is a $[p/4]$-dimensional subspace of self-dual models.

The second half of this paper is devoted to extending the results of Elitzur *et al* (1979) on the existence of a massless phase to the case of a general interaction. For $p \geq 5$, we find that there is always a $[p/2]$-dimensional region in which the model is in a massless phase. We establish lower bounds on the extent of this region. For the purposes of illustration we consider $p = 5, 6$. The case $p = 4$ is isomorphic to the Ashkin and Teller (1943) model. The consequences of this have been explored by Kadanoff (1977).

All that we have to say concerning duality properties applies equally to Z_p lattice gauge theories in four dimensions (Elitzur *et al* 1979). In fact, the equations are identical after replacing 'site' by 'link' and 'link' by 'plaquette.' Also, we expect, for large enough p, to find a region in the parameter space in which the models behave like free electrodynamics at large distances. A determination of the critical value of p awaits an accurate estimate of the critical temperature of the corresponding O(2) model, however.

Finally, in the appendix, we show that the massless phase also exists for the same model defined on a triangular or a honeycomb lattice, and that the condition on p for this to occur is probably once again $p \geq 5$.

2. Duality transformations

The models we consider are described by a partition function

$$Z = \operatorname*{tr}_{\theta_i} \exp[\Sigma_{\langle ij\rangle} V(\theta_i - \theta_j)] \tag{2.1}$$

where $\Sigma_{\langle ij\rangle}$ implies a sum over nearest-neighbour pairs. We write

$$x_r = \exp[V(2\pi r/p)] \tag{2.2}$$

so that each possible model (at a given temperature, $\beta = 1/kT$ being absorbed into the definition of V) corresponds to a point in the real projective space $\{x_r\}$, where all points $\lambda\{x_r\}$ (λ arbitrary) are identified. Since x_r has period p it can be Fourier-decomposed.

$$x_r = p^{-1/2} \sum_s \tilde{x}_s \exp(2\pi i r s/p) \tag{2.3}$$

where $\tilde{x}_s$ also has period p, so that the sum over s is over one period of the function. The factor $p^{-1/2}$ is inserted for convenience. (2.3) can be written in a real form (since x_r is an even function).

$$x_r = \sum_{s=0}^{[p/2]} D_{rs} x_s \qquad (r = 0, 1, \ldots, [p/2]) \tag{2.4}$$

where

$$\begin{aligned} D_{rs} &= 2p^{-1/2} \cos(2\pi rs/p) \qquad (s \neq 0, p/2) \\ D_{r0} &= p^{-1/2} \\ D_{r,p/2} &= p^{-1/2}(-1)^p. \end{aligned} \tag{2.5}$$

Inserting the Fourier decomposition (2.3) into (2.1), so that a variable $s_{ij} = -s_{ji}$ is assigned to each link $\langle ij\rangle$, it is possible to perform the trace over the θ_i, which gives zero unless the system of constraints

$$\sum_j s_{ij} = 0 \qquad (\mathrm{mod}\, p) \tag{2.6}$$

is satisfied. This can be done by defining integer variables $n_a = 0, 1, \ldots, (p-1)$ on the sites a of the dual lattice and letting $s_{ij} = n_a - n_b$ (mod p) whenever the link $\langle ij\rangle$ intersects the dual link $\langle ab\rangle$. We end up with a model defined on the dual lattice which differs from the original model only in that x_r is replaced by $\tilde{x}_r$. Thus the duality transformation is a *linear* transformation in the projective space $\{x_r\}$.

If we repeat the duality transformation we obtain the original model. This is expressed by the fact that, with our normalisation, $\boldsymbol{D}^2 = \mathbf{1}$.

We now look for self-dual models. These will correspond to eigenvectors of $\boldsymbol{D}$ with positive eigenvalues. Clearly all eigenvalues of $\boldsymbol{D}$ must be ± 1. Although $\boldsymbol{D}$ is not symmetric, because $\boldsymbol{D}^2 = \mathbf{1}$ any generalised eigenvector is also a true eigenvector. Since the generalised eigenvectors span the space, it follows that $\boldsymbol{D}$ is diagonalisable. We can also calculate the trace:

$$\mathrm{tr}\, \boldsymbol{D} = \begin{cases} 1 & ([p/2] \text{ even}) \\ 0 & ([p/2] \text{ odd}) \end{cases} \tag{2.7}$$

Hence the eigenvalue $+1$ of $\boldsymbol{D}$ must be $([p/4]+1)$-fold degenerate. There is, therefore, a $[p/4]$-dimensional linear subspace of self-dual models.

Two points of this subspace are already known. They are the Potts model, whose self-dual point corresponds to

$$x_r/x_0 = (1+\sqrt{p})^{-1} \qquad (r = 1, 2, \ldots, [p/2]) \tag{2.8}$$

and the Villain form

$$x_r = \sum_{n=-\infty}^{\infty} \exp[-2\pi^2 K(n + r/p)^2] \tag{2.9}$$

whose self-dual point lies at $K = p/2\pi$. For $4 \leqslant p \leqslant 7$, therefore, the general self-dual model is a linear combination of these two models in x_r space.

The parameter spaces x_r/x_0 for the cases $p = 4$, 5 and 6 are shown in figures 1, 2 and 3 respectively. The restrictions $x_r \geqslant 0$ implied by the existence of a dual model impose linear constraints on the x_r/x_0. Thus the allowed region is bounded by lines (planes) for $p = 4$, 5, (6). Those boundaries which lie within the hypercube $0 \leqslant x_r/x_0 \leqslant 1$ do not appear to correspond to phase boundaries—at these points the interactions of the dual model merely become imaginary. On the other hand, models with $x_r/x_0 > 1$ are antiferromagnetic in nature and should be in a quite different universality class. We do not discuss this further.

Some observations on figures 1–3 are in order. The case $p = 4$ is isomorphic to the Ashkin–Teller model, as can be seen by making the transformation

$$e^{i\theta_j} = (s_j + i\sigma_j)/(1+i)^{-1} \tag{2.10}$$

where $s_j, \sigma_j = \pm 1$ are Ising spins defined at each site. In the $(x_1/x_0, x_2/x_0)$ plane the critical line of the eight vertex model corresponds to the lower half of the self-dual line, terminating at the Potts self-dual point. In addition, for $x_1/x_0 = 0$, the model reduces to an Ising model, since either all $\theta_i = 0$ (mod π), or all $\theta_i = \pi/2$ (mod π). Lin and Wu

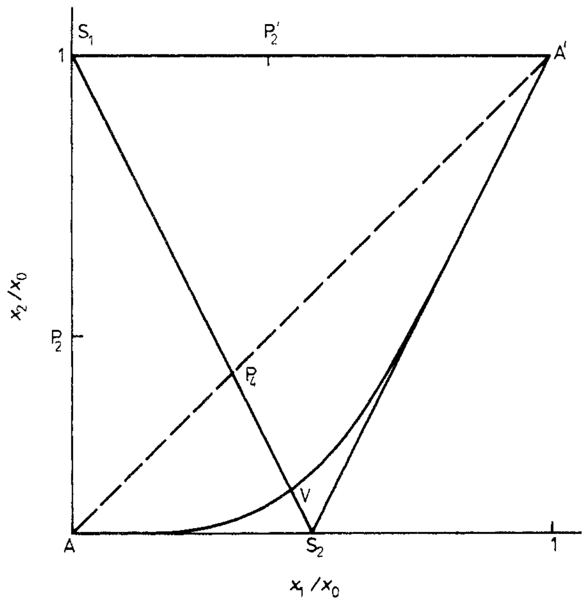

Figure 1. Parameter space for $p = 4$. A and A′ denote the low- and high-temperature limits, respectively. S_1S_2 is the self-dual line. Under duality AS_1S_2 is mapped into $A'S_1S_2$. The four-state Potts model corresponds to the line AA′, with a transition at P_4. P_4S_2 is the line of second-order transitions of the Ashkin–Teller model. In addition there are two lines of Ising transitions, originating at P_2 and P_2'. The Villain model corresponds to the curved line, with a transition at V.

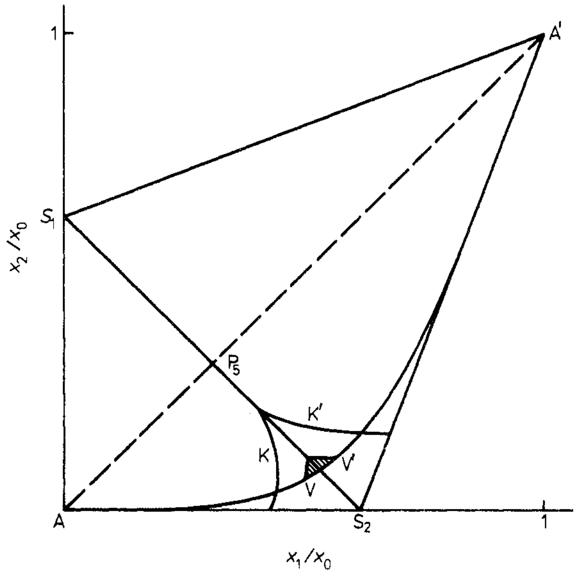

Figure 2. Parameter space for $p = 5$. Notation as in figure 1. In this case the phase diagram is symmetric about AA′. Shaded area corresponds to the lower bound on the extent of the massless phase proven in the text. K and K′ illustrate the conjectured approximate location of the two Kosterlitz–Thouless transitions.

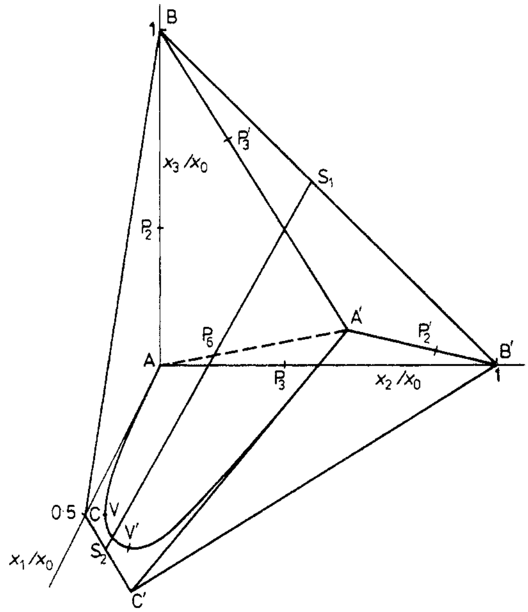

Figure 3. Parameter space for $p = 6$. Notation as in figure 1. The lower bounds on the massless region (now three-dimensional) are not shown but they form two mutually dual surfaces originating at the points V and V′ on the Villain curve, which converge and intersect in a curve which crosses the self-dual line. There are also (not shown) two surfaces of Ising transitions, originating at P_2 and P_2', and two surfaces of three-state Potts transitions, originating at P_3 and P_3'.

(1974) have conjectured the existence of a line of Ising-like transitions, beginning at the Ising point $x_2/x_0 = (1+\sqrt{2})^{-1}$ and terminating at the Potts self-dual point. There is also a second line of Ising-like transitions, dual to the first, beginning at $x_1/x_0 = (1+\sqrt{2})^{-1}$, $x_2/x_0 = 1$. This phase structure can be expressed in terms of the order parameters:

$$M_n = \langle e^{in\theta} \rangle \tag{2.11}$$

A model for which $x_2 < x_1$ (at a given temperature) undergoes one transition from the high-temperature phase in which $M_1 = M_2 = 0$ to the low-temperature phase in which both are non-zero. On the other hand, models with $x_2 > x_1$ undergo two successive Ising-like transitions, with an intermediate phase in which $M_1 = 0$ and $M_2 \neq 0$, leaving a residual Z_2 symmetry.

For $p = 5$ the phase diagram is symmetric under interchanging x_1 and x_2. This can be seen by making the transformation $\theta_i \to 2\theta_i$ (mod 2π). Such symmetries will always be present whenever the group Z_p has a non-trivial automorphism (excluding $\theta_i \to -\theta_i$).

For $p \geq 5$ there is a unique transition along the Potts line (Hintermann *et al* 1978) which is, moreover, first order (Baxter 1973). On the other hand, Eliztur *et al* (1979) have shown that along the Villain line (parametrised by (2.9) and illustrated in figures 1–3 there are two Kosterlitz–Thouless (1973) transitions for $p \geq 5$. One of the aims of the next section will be to gain information on the region between the Potts line and the Villain line.

For $p = 6$ there are, in addition, at least four other critical surfaces, similar to the Ising-like transitions present when $p = 4$. They originate in the Ising transition at

$x_1 = x_2 = 0$, $x_3/x_0 = (1+\sqrt{2})^{-1}$ and the three-state Potts model transition at $x_1 = x_3 = 0$, $x_2/x_0 = (1+\sqrt{3})^{-1}$, and their respective dual images. Thus, for suitable values of the couplings, the passage from the high-temperature phase to the low-temperature phase may occur by a three-state Potts transition followed by an Ising transition (with an intermediate phase in which $M_2 \neq 0$), or the transitions may occur in the reverse order, with an intermediate phase characterised by $M_3 \neq 0$. All of these surfaces lie on the other side of the Potts line from the Villain line and, since there is a unique transition along the Potts line, they are presumably unconnected with the two Kosterlitz–Thouless transitions.

3. The massless phase

In order to prove rigorous results for these models in the form of Griffiths inequalities, it is first necessary to choose a representation for the interaction which is amenable to the known techniques. Two such representations suggest themselves (we restrict ourselves to the case $p = 5$ for clarity):

$$x_r = \exp[K_1' \cos(2\pi r/5) + K_2' \cos(4\pi r/5)] \tag{3.1}$$

$$x_r = \sum_{n_1} \exp[-2\pi^2 K_1(n_1 + r/5)^2] \sum_{n_2} \exp[-2\pi^2 K_2(n_2 + 2r/5)]. \tag{3.2}$$

For general p, there are $[p/2]$ terms in each representation. (3.1) is, of course, completely general. However, it is possible to prove Griffiths-type inequalities using (3.1) only when K_1' and K_2' are both non-negative. On the other hand, (3.2) converges only when $K_1, K_2 > 0$, a region which is bounded by the Villain line ($K_2 = 0$) and its image ($K_1 = 0$) in the Potts line. This region contains that in which the K_i' of (3.1) are non-negative. Thus (3.2), the 'generalised Villain representation', is more useful.

We begin by summarising the arguments of Elitzur *et al*, which establish the existence of two phase transitions with an intermediate massless phase transitions with an intermediate mass less phase, for the pure Villain form ($K_2 = 0$).

Consider the correlation function in the Villain Z_p model at 'temperature' K_1^{-1}

$$\begin{aligned} g_q(\boldsymbol{R}; Z_p(K_1^{-1})) &= \langle \exp(\mathrm{i}q\theta(\boldsymbol{R})) \exp(-\mathrm{i}q\theta(\boldsymbol{O})) \rangle \\ &= Z^{-1} \operatorname{tr}_{\theta_i n_{ij}} \exp(\mathrm{i}q\theta(\boldsymbol{R}) - \theta(\boldsymbol{O})) \exp[-\tfrac{1}{2}K_1 \Sigma_{\langle ij \rangle} (\theta_i - \theta_j - 2\pi n_{ij})^2] \end{aligned} \tag{3.3}$$

where Z denotes the partition function. Elitzur *et al* (1979) establish upper and lower bounds on this correlation function, in terms of those of the O(2) model, obtained from (3.3) by allowing the θ_i to be continuous variables, and of the roughening model, obtained from (3.3) by dropping the term $2\pi n_{ij}$ (and allowing the θ_i to range over all multiples of $2\pi/p$ from $-\infty$ to $+\infty$). Thus:

$$g_q(\boldsymbol{R}; Z_\infty(K_1^{-1})) \leqslant g_q(\boldsymbol{R}, Z_p(K_1^{-1})) \leqslant g_q(\boldsymbol{R}, \text{roughening}) \tag{3.4}$$

where the right-hand side is explicitly

$$Z^{-1} \operatorname*{tr}_{r_i} \exp[(2\pi \mathrm{i}q/p)(r_i - r_j)] \exp[-(2\pi^2 K_1/p^2) \Sigma_{\langle ij \rangle} (r_i - r_j)^2] \tag{3.5}$$

where the r_i take all integer values.

Now, if $K_1^{-1} < T_c$, the critical temperature of the Villain XY model, the left-hand side is power-behaved as $|\boldsymbol{R}| \to \infty$. Similarly, if $p^2/4\pi^2 K_1 > T_R$, the critical temperature of the roughening model, the latter is in its rough phase and the right-hand side of (3.4) goes to zero as $|\boldsymbol{R}| \to \infty$. Now the Villain XY and the roughening models are mutually dual (José *et al* 1977), with $T_R = 1/T_c$. Thus the Z_p correlation function exhibits the behaviour characteristic of a massless phase if

$$4\pi^2 T_R/p^2 < K_1^{-1} < 1/T_R. \tag{3.6}$$

These bounds give upper and lower bounds, respectively, for the transition temperatures into the massless phase. (3.6) only applies, of course, if

$$p > 2\pi T_R. \tag{3.7}$$

Thus there will be three phases if $p > p_c$, where $p_c < 2\pi/T_R$. Elitzur *et al* estimated $T_c = T_R^{-1}$, but it is possible to obtain an upper bound for T_R in terms of the critical temperature $(2\ln 2)^{-1}$ of the F model (R H Swendsen, private communication). So $p_c < \pi/\ln 2 = 4{\cdot}53, \ldots$, and there are certainly three phases for $p \geqslant 5$.

We now extend these results into the region $K_2 > 0$. We consider the same correlation function $g_1(\boldsymbol{R}, K_1, K_2)$, where we have made the dependence on K_1 and K_2 explicit. By arguments very similar to those described by Elitzur *et al* it can be shown that

$$\frac{\partial}{\partial K_2} g_q(\boldsymbol{R}, K_1, K_2) \geqslant 0 \tag{3.8}$$

and so

$$g_q(\boldsymbol{R}, K_1, K_2) \geqslant g_q(\boldsymbol{R}, K_1, 0) \qquad \text{for } K_2 > 0. \tag{3.9}$$

K_2^{-1} can be regarded as a generalised temperature, and (3.8) expresses the fact that decreasing the temperature increases the correlations. The opposite inequality is

$$\frac{\partial}{\partial K_2} g_q(\boldsymbol{R}, \tilde{K}_1, \tilde{K}_2) \leqslant 0 \tag{3.10}$$

where $(\tilde{K}_1, \tilde{K}_2)$ are the parameters of the model dual to that specified by (K_1, K_2). We now define a region in the $(x_1/x_0, x_2/x_0)$ plane as follows. From the point $K_1 = 2\ln 2$, $K_2 = 0$ construct the curve $K_1 = 2\ln 2$, $K_2 > 0$ until it intersects the self-dual line. Similarly construct the dual of this curve. The area enclosed between these two curves and the Villain curve then gives a lower bound on the extent of the massless phase, by (3.8) and (3.10). Note that the Potts line lies well outside this region. Similar considerations hold for arbitrary p, but the associated regions are difficult to represent. We have checked that there is never any overlap with the Potts line.

4. Conclusions

We have established the existence of a massless phase, similar to the low-temperature phase of the XY model, in the general Z_p models for $p \geqslant 5$. If these models are in the same universality class as the XY model with Z_p perturbations analysed by José *et al* (1977) the phase boundaries will correspond to infinite-order Kosterlitz–Thouless-type transitions. We have shown that the critical value p_c of p above which these two transitions occur is less than $4{\cdot}53$ and, according to the analysis of José *et al*, may well be exactly $p_c = 4$. It is very suggestive that this is also the critical value for p at which the Potts model transition becomes first order (Baxter 1973). If the same mechanism is responsible for both the onset of first-order transitions in the Potts model and the splitting of the second-order transition for $p = 4$ into two Kosterlitz–Thouless transitions for $p > 4$, then it is reasonable to conjecture that when the two infinite-order phase boundaries meet on the self-dual line (figure 2) the transition becomes first order, this first-order nature persisting all the way up to the Potts line. Thus, a conventional second-order transition between the high-temperature and low-temperature phase would be forbidden for $p \geqslant 5$. The system would have to go via a first-order transition, two Kosterlitz–Thouless transitions or successive Ising, three-state Potts, or Ashkin–Teller-like transitions. At the lower Kosterlitz–Thouless transition, the order parameters M_n will fall to zero faster than any power. A simple application of the renormalisation group equations of José *et al* (1977) leads to the prediction

$$M_n \propto \exp(-Cn^2/(T_c - T)^{1/2}) \tag{4.1}$$

where C is a non-universal constant.

Finally we mention that the results of this paper can be extended in a straightforward manner to models invariant under any finitely-generated Abelian group. No new physics emerges.

Acknowledgments

The author would like to thank R H Swendsen, R B Pearson and D R Nelson for discussions and communications.

Appendix

We extend the results on the existence of a massless phase to the triangular and honeycomb lattices. It might be thought that the result $p_c = 4$ for the square lattice is connected with the spatial Z_4 symmetry of that lattice. However, we show that p_c is close to 4 for the other lattices also. The inequality arguments of § 3 go through independent of the lattice, giving the lower bounds (similar to (3.6)) on the massless phase for the Villain form:

$$4\pi^2 T_R/p^2 < K_1^{-1} < T_c \tag{A1}$$

so that

$$p_c < 2\pi(T_R/T_c)^{1/2} \tag{A2}$$

where T_c and T_R are the transition temperatures of the Villain XY and the roughening models, respectively, for the given lattice. Since the honeycomb (hc) and triangular (t) lattices are mutually dual:

$$T_c^t T_R^{hc} = T_c^{hc} T_R^t = 1 \tag{A3}$$

so, for both lattices,

$$p_c < 2\pi(T_c^t T_c^{hc})^{-1/2}. \tag{A4}$$

In the absence of any rigorous lower bounds on T_c^t and T_c^{hc}, we shall use the estimate provided by the Kosterlitz–Thouless (1973) criterion. This involves beginning with the spin-wave limit of the Villain form:

$$Z = \operatorname*{tr}_{\theta_i} \exp\left[-(1/2T) \sum_{ij} (\theta_i - \theta_j)^2\right] \tag{A5}$$

and going to the continuum limit, which gives a reduced Hamiltonian:

$$H = (x/2T) \int (\nabla\theta)^2 \, \mathrm{d}^2 r \tag{A6}$$

where $x = \sqrt{3}$, $1/\sqrt{3}$, and 1 for the triangular, honeycomb and square lattices, respectively. Now the energy of a vortex is balanced against its entropy, giving the estimate

$$T_c \approx \pi x/2. \tag{A7}$$

We expect this to be an upper bound on T_c (due to screening by other vortices) so this gives $p_c \approx 4$.

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